Math 155, ***Lecture Notes- Bonds*** **Name____________**

Section 8.5 *Partial Fractions*

In this section we will consider the *method of partial fractions*, a technique for decomposing a rational function into simpler rational functions.

When the numerator is not the derivative of the denominator, nor is it a constant multiple of the derivative of the denominator, a simple substitution won't work. However, if we notice that the integrand can be decomposed, then the integral is actually quite simple:

$$\int \frac{x}{x^2-5x+6}dx = \int \left[\frac{-2}{x-2} + \frac{3}{x-3}\right]dx$$

$$= -2\int \frac{1}{x-2}dx + 3\int \frac{1}{x-3}dx = -2\ln|x-2| + 3\ln|x-3| + C$$

Ex.1 Decompose: $\frac{x}{x^2-5x+6}$

Ex.2 Integrate: $\int \frac{3x+11}{x^2-x-6}dx =$

Ex.3 Integrate: $\int \frac{1}{4x^2-9}dx =$

Ex.3 continued

Ex.4 Evaluate: $\int \frac{4x^2}{x^3+x^2-x-1}dx =$

Ex.4 continued

continued

continued

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \qquad \text{and} \qquad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Ex.5 FACTOR:

$$\begin{aligned}
&x^5 + x^4 - x - 1 \\
&= x^4(x+1) - (x+1) \\
&= (x^4 - 1)(x+1) \\
&= (x^2+1)(x^2-1)(x+1) \\
&= (x^2+1)(x+1)(x-1)(x+1) \\
&= (x-1)(x+1)^2(x^2+1)
\end{aligned}$$

Terminology: $(x-1)$ is called a *linear factor*,

$(x+1)^2$ is called a *repeated linear factor*, and

(x^2+1) is called an *irreducible quadratic factor*.

Ex.6 Use Long Division, then Integrate: $\int \frac{x^3 - x + 3}{x^2 + x - 2} dx$

Ex.6 continued

Ex.7 Evaluate: $\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$

Ex.7 continued

Ex.8 Evaluate: $\int \frac{x^2+x+3}{x^4+6x^2+9}dx$

Ex.8 continued

Ex.9 Decompose: $\dfrac{3x-5}{x^3-1}$

Ex.9 continued

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the roots of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.